THE LOGIC OF COMPARATIVE THEORY EVALUATION

by

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ABSTRACT

Schaffner's logic of comparative theory evaluation is criticised for an inappropriate analysis of ad hocness. An alternative analysis, based on Zahar's account of novelty, is given and extended to the case of multiple successful predictions by a theory. The application of the method to the appraisal of quantitative prediction is discussed.

The Logic of Comparative Theory Evaluation *

1. The Bayesian Analysis of Ad hocness.

In a recent note Schaffner ([1974]) has given a formal discussion of the notion of ad hocness in terms of a Bayesian model for the appraisal of theories. Schaffner develops his general ideas in the context of a critique of Zahar's [1973] which was concerned with the particular problem of comparing the Einstein and Lorentz research programmes. Zahar suggests the following analysis of ad hocness[1]:

> Ad hoc_1: A theory is said to be ad hoc_1 if it has no novel consequences as compared with its predecessor.
>
> Ad hoc_2: [A theory]... is ad hoc_2 if none of its novel predictions have been actually 'verified'.
>
> Ad hoc_3: [A]... theory is said to be ad hoc_3 if it is obtained from its predecessor through a modification of the auxiliary hypotheses which does not accord with the spirit of the heuristic of the programme.

Zahar explains the meaning of novelty as follows[2]

> A fact will be considered novel with respect to a given hypothesis if it did not belong to the problem-situation which governed the construction of the hypothesis.

Schaffner begins his elucidation by discussing the notions of ad hoc_1 and ad hoc_3. The first he describes as a logical dream, since the novel consequences of a theory cannot in practice be "surveyed", so the question of whether a theory is ad hoc_1 can only be discussed relative to the extent to which novel consequences have been looked for at the particular epoch of the evaluation. Ad hoc_3 Schaffner claims to be "vague to the point of inapplicability". For Schaffner ad hoc_2 is "close to

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the sense in which ad hoc is used in science" and he announces that his Bayesian analysis will be brought to bear on this sense of ad hoc.

Denoting by $p(T/b \,\&\, e)$ the probability of a theory T to be true in the light of background knowledge b and the positive outcome e of some experiment not part of b, by $p(T/b)$ the prior assessment of T, by $p(e/T \,\&\, b)$ the probability of obtaining e given T and b, and by $p(e/b)$ the probability of obtaining e on the basis of background knowledge alone, Schaffner writes

$$p(T/b \,\&\, e) = \frac{p(T/b).p(e/T \,\&\, b)}{p(e/b)} \qquad (1)$$

If T explains e we can set $p(e/T \,\&\, b) = 1$ so in this case we obtain

$$p(T/b \,\&\, e) = p(T/b) \big/ p(e/b) \qquad (2)$$

Schaffner proceeds to discuss ad hocness as a property of an hypothesis as a constituent part of a theory, but in order to keep the argument as simple as possible we shall follow Zahar in considering the ad hocness of theories. Translated into these terms Schaffner's idea is that a theory T gives an ad hoc explanation of an experimental result e if $p(T/b)$ is close to zero and $p(e/b)$ is significantly larger than $p(T/b)$. The argument for $p(T/b)$ being small is that it has no "theoretical support" (or indeed empirical support other than e itself). This looks suspiciously like a reference to ad hoc_3, and Schaffner now appears to be claiming that a theory is ad hoc_2 partly in virtue of its being ad hoc_3. So he is not really giving an independent analysis of ad hoc_2 at all.

The lack of novelty in the prediction of e is associated by Schaffner with a high value for $p(e/b)$. On Zahar's account this is a necessary condition for lack of novelty, but not a sufficient condition. We proceed to show how using Zahar's notion

of novelty one can get an "internal" Bayesian analysis of ad hoc_2.

We express Bayes's theorem in the following familiar way

$$p(T/b\,\&\,e) = \frac{p(T/b).\,p(e/T\,\&\,b)}{p(e/T\,\&\,b)p(T/b) + p(e/{\sim}T\,\&\,b).\,p({\sim}T/b)} \qquad (3)$$

where ${\sim}T$ denotes the negation of T.

We follow Schaffner in interpreting p(A/B) as the degree of belief that A is true given that B is true. Note also that e is supposed in equation (3) to refer to a prediction made by the theory T (we could perhaps write e_T to emphasize this important point) so $p(e/{\sim}T\,\&\,b)$ means the probability that the prediction e_T derived from the theory T is a true prediction given that the background information b is true but the theory T is false (i.e., its consequences are not guaranteed to be true although "by accident" they may be true).

Writing $p(T/b) = x$, $p(e/{\sim}T\,\&\,b) = \varepsilon$

and taking $p(e/T\,\&\,b) = 1$ and using $p({\sim}T/b) = 1 - x$

$$p(T/b\,\&\,e) = \frac{x}{x + \varepsilon(1 - x)} \qquad (4)$$

We define an enhancement ratio γ by

$$\gamma = \frac{p(T/b\,\&\,e)}{p(T/b)}$$

whence using (4) we obtain the simple result

$$\gamma = \frac{1}{x + \varepsilon(1 - x)} \qquad (5)$$

We can now explain that if a theory T is ad hoc_2 with respect to the experiment e then $\varepsilon = 1$, i.e. the explanation of e by T in no way depends on the truth or falsehood of T, both of which eventualities lead with certainty to the result e. This is just what a scientist means when he says T was an ad hoc explanation of e, namely T was devised for the express purpose of explaining e,

so the explanation of e is guaranteed independent of whether T is true or false. To show the consistency of our analysis if we put $\varepsilon = 1$ in (5) we get $\gamma = 1$, so the posterior and prior probabilities of T are equal (there is no enhancement) and this again is just what we expect from an ad hoc explanation of e, namely e itself gives us no additional information for assessing the truth of T.[3]
Notice that $p(e/b) = x + \varepsilon(1 - x)$ is equal to unity if $\varepsilon = 1$, but that $p(e/b) = 1$ is achieved for $x = 1$ whatever the value of ε, so the explication of novelty in terms of low $p(e/b)$ (Schaffner) and small ε (Zahar) are by no means equivalent.

On our analysis if $x \ll \varepsilon \ll 1$, then $\gamma \simeq 1/\varepsilon$, so in this case we get a big enhancement and the theory is far from being ad hoc. Noting that under these conditions $p(e/b) \simeq \varepsilon$, we see that this is a situation in which Schaffner would claim that the theory was ad hoc, which highlights the way in which his analysis differs from ours. Effectively Schaffner requires γx to be small as his condition of ad hocness. He is thus concerned with the absolute value of the probability of a theory after it has explained some experimental finding. If this absolute value is still small the theory is to be regarded as an ad hoc explanation of the experiment. On our account the important aspect in assessing ad hocness for this case is not the absolute value of the probability but the enhancement ratio. Our point against Schaffner is not that his analysis may not explicate some legitimate sense of the ambiguous appellation ad hoc, but that it fails to explicate Zahar's very important notion of ad hoc_2. There is no inconsistency here on Schaffner's part, since he finds Zahar's account of ad hoc_2 inadequate in respect of the definition of novelty involved with its [redacted]-historical associations, but we would maintain that Zahar's sense of ad hoc_2 is the one that ought to be explicated since it is the one most importantly used in science.

2. The Case of Multiple Predictions

To develop our analysis a little further we can consider how a theory builds up a favourable appraisal as it makes a number of successful predictions $e_1, e_2 \ldots e_n$ say. Denote by p_s the posterior probability $p(T/b \& e_1 \& e_2 \ldots \& e_s)$ after s successful predictions. Assuming the predictions are quite independent and for simplicity are all associated with the same value of ε, we can clearly write

$$p_n = \gamma^{(n)} \times \gamma^{(n-1)} \times \ldots \times \gamma^{(1)} . p_o$$

where $p_o = x$ according to our previous notation

and $$\gamma^{(s)} = \frac{1}{p_{s-1}(1-\varepsilon) + \varepsilon}$$

$$p_{s+1} = \gamma^{(s+1)} . p_s$$

The solution of this recursion is by inspection or more simply by replacing ε by ε^n in the formula for p_1 (see (4) above). We obtain[4]

$$p_n = \frac{1}{1 - \varepsilon^n + \varepsilon^n/x} \qquad (6)$$

We can also ask what is the probability for the $(n+1)^{th}$ prediction e_{n+1} being correct if the theory has already made n successful predictions $e_1 \ldots e_n$. Denoting this probability by $p(e\,n+1)$ we clearly obtain the result

$$p(e\,n+1) = \varepsilon + \frac{1-\varepsilon}{1 - \varepsilon^n + \varepsilon^n/x} \qquad (7)$$

If ε is a small quantity (i.e. $\ll 1$) which will be the case if T is non-ad hoc_2 with respect to all the predictions, we can write the following formulae which will be perfectly satisfactory for the subsequent discussion

$$p_n \simeq \frac{1}{1 + \varepsilon^n/x}, \qquad (8)$$

$$p(e_{n+1}) \simeq \mathcal{E} + p_n \qquad (9)$$

$$\gamma^{(n)} \simeq \frac{1}{\mathcal{E} + p_{n-1}} \qquad (10)$$

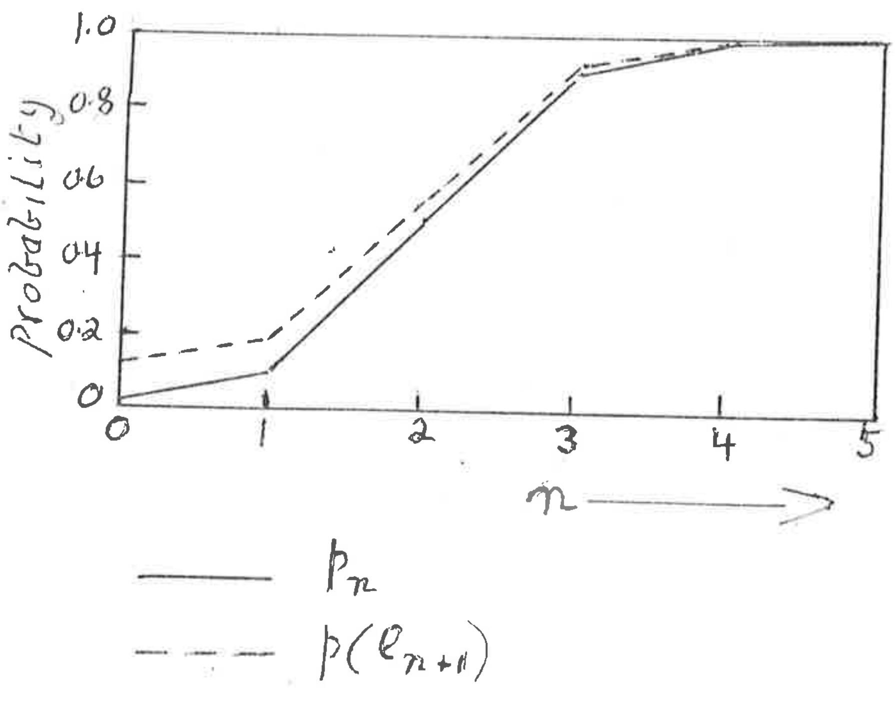
We note the following features

(1) The value of pn depends entirely on the ratio $\mathcal{E}^n/x$
For $\mathcal{E}^n/x \gg 1$, $p_n \ll 1$
and for $\mathcal{E}^n/x \ll 1$, $p_n \simeq 1$
At the critical point $\mathcal{E}^n = x$, we have $p_n \simeq 1/2$.
So if initially $x \ll \mathcal{E}$ as n increases pn will rise steeply as we reach the value $n = \ln x/\ln \mathcal{E}$.

(2) So long as $p_n \ll \mathcal{E}$, we have $p(\ell_{n+1}) \simeq \mathcal{E}$, but as pn builds up towards unity, so does $p(\ell_{n+1})$.

(3) So long as $p_{n-1} \ll \mathcal{E}$, $\gamma^{(n)} \simeq 1/\mathcal{E}$, but as p_{n-1} builds up towards unity the enhancement factor $\gamma^{(n)}$ also tends to unity.

To take a concrete case we illustrate in the accompanying figure p_n and $p(\ell_{n+1})$ as functions of n for the particular choice $x = 0.01$, $\mathcal{E} = 0.1$.



of n this condition will fail. For example in the case of the anomalous magnetic moment of the electron the effect of hadronic couplings introduce theoretical uncertainties which would ultimately make the prediction of the theory unreliable.[9]

4. Conclusion

The concept of a novel prediction plays a very important part in the way scientists assess their confidence in theories. Of course on a purely instrumentalist view of theories the question of novelty in Zahar's sense is of no consequence. But to a realist it is only in virtue of novel successful predictions that commitment to a theory can find rational justification. Our analysis has shown how the logic of comparative theory evaluation can be viewed if degrees of commitment are governed by Bayesian rationality constraints.

FOOTNOTES

1. Zahar [1973] p.101. In his [1974] Zahar rephrases his definition in terms of a notion of empirical non-ad hocness expressed as a three-place relation between an observation statement, a theory, and a heuristic. See also in this connection the very clear account of empirical support given by Worrall in his [1975]. However this more recent work has somewhat obscured the important distinctions drawn by Zahar in his [1973].

2. Zahar [1973] p.103.

3. What we have shown is that $\varepsilon = 1$ is a necessary condition for T to be an ad hoc_2 explanation of e. To justify $\varepsilon = 1$ as a sufficient condition we must invoke a principle of insufficient reason, viz., if there is no reason for the community of scientists to entertain with non-vanishing prior probability only theories which explain e, then they will not so constrain their choice of alternative theories.

4. The successful detailed quantitative predictions of a theory in respect of phenomena quite different from those which the theory was originally proposed to deal with have always attracted the attention of scientists. To take an example at random, in referring to his early work on the ground-state of Helium Hylleraas comments in his [1963] (p.42) "The end result of my calculation was...greatly admired and thought of as almost a proof of the validity of wave mechanics...in the strict numerical sense".

5. Van Dyck et al. ([1977]).

6. See value quoted in Calmet et al. ([1977]). For a good account of the fluctuating agreement between theory and experiment the reviews by Lautrup, Peterman and de Raphael ([1972]) or Rich and Wesley ([1972]) may be consulted.

7. We may refer to Shimony's concept of commitment to a theory (see his [1970] pp.94-95). The degree of commitment measures the scientist's belief that the theory T belongs to the equivalence class of all theories which give the same true observational predictions within its domain of current experimentation and that the "true" theory "generalizes" in some sense the concepts embodied in T. Commitment measures our belief, not that a theory is true, but that it points the way to the truth. Naturally this second aspect of commitment cannot be assessed via novel predictions within the current experimental domain.

8. It is easy to see that our confidence in a theory at a given level of accuracy for agreement between theory and experiment does not depend on the particular scale of notation used to express the result.

9. According to Rich and Wesley ([1972]) the known hadronic contribution to the electron anomaly would affect the tenth significant figure.

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